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Abstract

We give geometrical conditions under which there exist extremal functions for the sharp L^2-Nash inequality.

1 Introduction

This paper is in the spirit of several works on best constants problems in Sobolev type inequalities. A general reference on this subject is the recent book of Hebey [9]. These questions have many interests. At first, they are at the origin of the resolution of famous geometrical problems as Yamabe problem. More generally, they show how geometry and analysis interact on Riemannian manifolds and lead to the developpement of interesting analytic methods. This article is devoted to the existence of extremal functions for the optimal L^2-Nash inequality and follows another paper [10] in which we proved the existence of a second best constant in the L^2-Nash inequality. Obviously, finding extremal functions is interesting from PDEs' point of view. The proof we give here may appear very technical. Nevertheless, its interest lies in the analytic methods it gives, for example on what concerns the study of concentration phenomenons. Moreover, extremal functions have their own interests because they give informations on best constants. For example, the existence of extremal functions for the circle S^1 gives an explicit inequality on S^1 (see [10]).

In this paper, we let (M,g) be a smooth compact Riemannian n-manifold. We consider the following inequality : for $u \in C^\infty(M)$,

$$(\int_M u^2 dv_g)^{1+\frac{2}{n}} \leq (A\int_M |\nabla u|_g^2 dv_g + B\int_M u^2 dv_g)(\int_M |u| dv_g)^{\frac{4}{n}} \qquad N(A,B)(u)$$

We say that $N(A,B)$ *is valid* if $N(A,B)(u)$ is true for all $u \in C^\infty(M)$. In the following, we refer to this inequality as the L^2-Nash inequality. Let now

$$A_0 = \inf\{A>0 | \text{ there exists } B>0 \text{ s.t. } N(A,B) \text{ is valid }\}$$

It was shown in [3] that

$$A_0 = A_0(n) = \frac{(n+2)^{\frac{n+2}{n}}}{2^{\frac{2}{n}} n\lambda_1(\mathcal{B}) |\mathcal{B}|^{\frac{2}{n}}}$$

where $|\mathcal{B}|$ is the volume of the unit ball $\mathcal{B}$ in $\mathbb{R}^n$, λ_1 is the first nonzero Neumann eigenvalue of the Laplacian for radial functions on $\mathcal{B}$ and $Vol(M)$ is the volume of (M,g). Then, it was shown in [10] that there exists $B>0$ such that the sharp $N(A_0(n),B)$ is valid. Another form of sharp inequality is in Druet-Hebey-Vaugon [6]. Let now

$$B_0 = \inf\{B \in \mathbb{R} \text{ s.t. } N(A_0(n),B) \text{ is valid }\}$$

It was also proved in [10] that for any smooth compact Riemannian n-manifold (M,g),

$$B_0 \geq \max\left(Vol(M)^{-\frac{2}{n}}, \frac{|\mathcal{B}|^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2}+\frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}}\max_{x\in M} S_g(x)\right)$$

where $S_g(x)$ is the scalar curvature of g at x. We now say that $u \in H_1^2(M)$, $u \not\equiv 0$ is an extremal function for the sharp L^2- inequality $N(A_0(n), B_0)$ if

$$\left(\int_M u^2 dv_g\right)^{1+\frac{2}{n}} = \left(A_0(n)\int_M |\nabla u|_g^2 dv_g + B_0\int_M u^2 dv_g\right)\left(\int_M |u|\, dv_g\right)^{\frac{4}{n}}$$

Such a study was carried out for sharp Sobolev inequalities by Djadli and Druet in the very nice reference [4]. Though they are close in their statement, these two questions, to know whether or not there exist extremal functions for sharp Sobolev inequalities and for the sharp L^2-Nash inequality, are however distinct in nature. In consequence, the problems we have to face here are very different from the one that appears in [4]. The main result of this article is the following :

Theorem 1 *Let (M,g) be a smooth compact Riemannian n-manifold. Let also B_0 be as above. Assume that :*

$$B_0 > \frac{|\mathcal{B}|^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2}+\frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}}\max_{x\in M} S_g(x)$$

Then, there exist extremal functions of class $C^{1,a}(M)$ ($0<a<1$) for the sharp L^2-Nash inequality.

We present here the main ideas of the proof of this theorem which is based on a precise study of a phenomenom of concentration. Namely, for $B < B_0$, we prove the existence of an extremal function u_B for inequality $N(A_B, B)$ where

$$A_B = \inf\{A| \text{ s.t. } N(A,B) \text{ is true }\} > A_0(n)$$

We then let $B \to B_0$. Standard theory shows that there exists $u \in H_1^2(M)$ such that $u_B \to u$ weakly in $H_1^2(M)$ when $B \to B_0$. We have to consider two cases. First, if $u \not\equiv 0$, it is not difficult to prove that u is an extremal function for $N(A_0(n), B_0)$. If $u \equiv 0$, we prove that u concentrates around a point x of M. In other words, $u_B \to 0$ when $B \to B_0$ in $C^0_{loc}(M - \{x\})$ and for all $\delta > 0$,

$$\lim_{B\to B_0} \frac{\int_{B(x,\delta)} u_B^2 dv_g}{\int_M u_B^2 dv_g} = 1$$

Hence, if η is a cut-off function such that $\eta \equiv 1$ in a neighbourhood of x and $\eta \equiv 0$ on $M - B(x,\delta)$ where δ is small, ηu_B have almost the same properties than u_B. Via exponential map at x, ηu_B can be seen as a function on $\mathbb{R}^n$ on which we have the standard optimal Nash inequality

$$\left(\int_{\mathbb{R}^n} (\eta u_B)^2 dx\right)^{1+\frac{2}{n}} \leq A_0(n)\int_{\mathbb{R}^n} |\nabla \eta u_B|^2 dx \left(\int_{\mathbb{R}^n} |\eta u_B|dx\right)^{\frac{4}{n}}$$

With the use of Cartan's expansion of the metric around x and precise estimations of the concentration of u_B, these integrals can be compared to the corresponding integrals on (M,g). We get that

$$\int_M (\eta u_B)^N dv_g \leq \alpha_B$$

where α_B is an expression involving integrals of u_B. Thanks to the Euler equation of u_B, we get that

$$\alpha'_B \leq \int_M (\eta u_B)^N dv_g$$

where α'_B is another expression involving integrals of u_B. The inegality $\alpha'_B \leq \alpha_B$ leads to

$$B_0 \leq \frac{\mid \mathcal{B} \mid^{-\frac{2}{n}}}{6n} \left(\frac{2}{n+2} + \frac{n-2}{\lambda_1} \right) \left(\frac{n+2}{2} \right)^{\frac{2}{n}} \max_{x \in M} S_g(x)$$

This gives the theorem.

As a consequence of theorem 1, we immediately have :

Corollary 1 *Let (M, g) be a smooth compact Riemannian n-manifold. We assume that*

$$Vol(M)^{-\frac{2}{n}} > \frac{\mid \mathcal{B} \mid^{-\frac{2}{n}}}{6n} \left(\frac{2}{n+2} + \frac{n-2}{\lambda_1} \right) \left(\frac{n+2}{2} \right)^{\frac{2}{n}} \max_{x \in M} S_g(x)$$

Then, there exist extremal functions of class $C^{1,a}(M)$ ($0 < a < 1$) for the sharp L^2-Nash inequality. In particular, this is the case if the scalar curvature is nonpositive.

For $n \geq 2$, the results obtained in [10] on the existence of extremal functions for the sharp L^2-Nash inequality are a consequence of theorem 1. For $n = 1$, we proved in [10] that constant functions are extremal functions for the sharp L^2-Nash inequality. At the moment, we are not able to give examples manifolds such that there does not exist extremal functions for the sharp L^2-Nash inequality. Hebey and Vaugon prove in [8] the existence of such manifolds in the case of Sobolev inequality. However, their proof strongly uses the conformal invariance of their inequality and we do not know yet some other methods to obtain this type of results.

2 Proof of theorem 1

Let $A_0(n)$ and B_0 be as in introduction. We define $\alpha_0 = B_0 A_0(n)^{-1}$. For $\alpha > 0$, we let also

$$\begin{aligned} I_\alpha(u) &= \frac{(\int_M \mid \nabla u \mid_g^2 dv_g + (\alpha_0 - \alpha) \int_M u^2 dv_g)(\int_M \mid u \mid^{1+\epsilon_\alpha} dv_g)^{\frac{4}{n(1+\epsilon_\alpha)}}}{(\int_M u^2 dv_g)^{1+\frac{2}{n}}} \\ \Lambda &= \{u \in C^\infty(M) \text{ s.t. } \int_M u^2 dv_g = 1\} \end{aligned}$$

and

$$\mu_\alpha = \inf_{u \in \Lambda} I_\alpha(u)$$

where ϵ_α is chosen such that

$$\lim_{\alpha \to 0} \epsilon_\alpha = 0, \ \mu_\alpha < A_0(n)^{-1} \text{ and, } \lim_{\alpha \to 0} \mu_\alpha = A_0(n)^{-1} \tag{1}$$

Clearly there exists $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$, such that

$$\int_M u_\alpha^2 dv_g = 1 \text{ and } \mu_\alpha = I_\alpha(u_\alpha)$$

We write now the Euler equation of u_α to get that, in the sense of distributions :

$$2A_\alpha \Delta_g u_\alpha + \frac{4}{n} B_\alpha u_\alpha^{\epsilon_\alpha} = k_\alpha u_\alpha \tag{E_α}$$

where Δ_g stands for the Laplacian with the minus sign convention and :

$$\begin{aligned}
A_\alpha &= \left(\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g\right)^{\frac{4}{n(1+\epsilon_\alpha)}} \\
B_\alpha &= \left(\int_M \mid \nabla u_\alpha \mid_g^{\,2} dv_g + (\alpha_0 - \alpha)\right)\left(\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g\right)^{\frac{4}{n(1+\epsilon_\alpha)}-1} \\
k_\alpha &= \frac{4}{n}\mu_\alpha + 2\int_M \mid \nabla u_\alpha \mid_g^2 dv_g \left(\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g\right)^{\frac{4}{n(1+\epsilon_\alpha)}}
\end{aligned}$$

By the Sobolev embedding theorem, $u_\alpha \in L^{\frac{2n}{n-2}}(M)$ and then, by classical methods, $u_\alpha \in C^2(M)$. To prove the theorem, we assume that there does not exists extremal functions for the sharp L^2-Nash inequality and show that

$$B_0 \leq \frac{\mid \mathcal{B} \mid^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2} + \frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}} \max_{x\in M} S_g(x)$$

As easily seen, the existence of extremal functions follows from an assumption like :

$$\liminf_{\alpha\to 0} \int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g > 0$$

Note that such an assumption implies that :

$$\int_M \mid \nabla u_\alpha \mid_g^{\,2} dv_g \leq C$$

In the following, we then assume that

$$\lim_{\alpha\to 0} \int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g = 0$$

or, equivalently :

$$\lim_{\alpha\to 0} A_\alpha = 0 \tag{2}$$

Now, using $N(A_0(n), B_0)(u_\alpha)$, we have :

$$\liminf_{\alpha\to 0} \int_M \mid \nabla u_\alpha \mid_g^{\,2} dv_g \Big(\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g\Big)^{\frac{4}{n(1+\epsilon_\alpha)}} \geq A_0(n)^{-1}$$

In addition, since $\mu_\alpha < A_0(n)^{-1}$, it is clear that :

$$\limsup_{\alpha\to 0} \int_M \mid \nabla u_\alpha \mid_g^{\,2} dv_g \Big(\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g\Big)^{\frac{4}{n(1+\epsilon_\alpha)}} \leq A_0(n)^{-1}$$

As a consequence, one easily checks that :

$$\lim_{\alpha\to 0} A_\alpha \int_M \mid \nabla u_\alpha \mid_g^{\,2} dv_g = A_0(n)^{-1} \tag{3}$$

$$\lim_{\alpha\to 0} B_\alpha \int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g = A_0(n)^{-1} \tag{4}$$

$$\lim_{\alpha\to 0} k_\alpha = (2+\frac{4}{n})A_0(n)^{-1} \tag{5}$$

The proof of the theorem proceeds in several steps. Step 1 to 4 are somehow similar than what was done in [10]. Note however that the limits are not anymore limits as $\alpha\to\infty$. Step 5 is a preparation to the concluding step, step 6.

We let $a_\alpha = {A_\alpha}^{\frac{1}{2}}$. We let also x_α be a point of M such that $u_\alpha(x_\alpha) = \parallel u_\alpha \parallel_\infty$. In the following, $B(p,r)$ denotes the ball of center p and radius r in $\mathbb{R}^n$ and $B_p(r)$ denotes the ball of center p and radius r in M. We assume in addition that bounded sequences are convergent, with no mention to the extracting of a subsequence, and write C for positive constants that do not depend on α.

Step 1 *For all* $\delta>0$ *:* $\liminf_{\alpha\to 0} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} > 0$

Let, for $x\in B(0,\delta)\subset\mathbb{R}^n$:

$$\begin{aligned} g_\alpha(x) &= (exp_{x_\alpha})^* g(a_\alpha x) \\ \varphi_\alpha(x) &= \parallel u_\alpha \parallel_\infty^{-1} u_\alpha(exp_{x_\alpha}(a_\alpha x)) \end{aligned}$$

We easily get :

$$\Delta_{g_\alpha}\varphi_\alpha + \frac{2}{n}\parallel u_\alpha \parallel_\infty^{-1+\epsilon_\alpha} B_\alpha \varphi_\alpha^{\epsilon_\alpha} = \frac{k_\alpha}{2}\varphi_\alpha \tag{$\tilde{E}_\alpha$}$$

Since $\Delta_g u_\alpha(x_\alpha)\geq 0$, we get from (E_α) and (5) :

$$\parallel u_\alpha \parallel_\infty^{\epsilon_\alpha} B_\alpha \leq C \parallel u_\alpha \parallel_\infty \tag{6}$$

and since $\parallel \varphi_\alpha \parallel_{L^\infty(B(0,\delta))}\leq 1$, we get from $(\tilde{E}_\alpha)$:

$$\parallel \Delta_{g_\alpha}\varphi_\alpha \parallel_{L^\infty(B(0,\delta))} \leq C$$

By classical methods, it follows that, for $a\in]0,1[$: $\parallel \varphi_\alpha \parallel_{C^{1,a}B(0,\delta)}\leq C$. Hence, $(\varphi_\alpha)_\alpha$ is equicontinuous and by Ascoli's theorem, there exists $\varphi\in C^0(B(0,\delta))$ such that $\varphi_\alpha\to\varphi$ in $C^0(B(0,\delta))$ as $\alpha\to 0$. We have :

$$\varphi(0) = \lim_{\alpha\to 0}\varphi_\alpha(0) = 1 \tag{7}$$

and also :

$$\begin{aligned} \int_{B(0,\delta)} {\varphi_\alpha}^{1+\epsilon_\alpha} dv_{g_\alpha} &= \parallel u_\alpha \parallel_\infty^{-(1+\epsilon_\alpha)} {A_\alpha}^{-\frac{n}{2}} \int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g \\ &= \parallel u_\alpha \parallel_\infty^{-(1+\epsilon_\alpha)} {A_\alpha}^{-\frac{n}{4}(1-\epsilon_\alpha)} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \\ &\leq \parallel u_\alpha \parallel_\infty^{-1} {A_\alpha}^{-\frac{n}{4}} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \end{aligned} \tag{8}$$

Since $\parallel u_\alpha \parallel_\infty^{\epsilon_\alpha}\geq 1$, (6) implies : $\parallel u_\alpha \parallel_\infty \geq C.B_\alpha$ and since $A_\alpha\to 0$ as $\alpha\to 0$, (4) implies that $B_\alpha \geq C.{A_\alpha}^{-\frac{n}{4}(1+\epsilon_\alpha)} \geq C.{A_\alpha}^{-\frac{n}{4}}$. Inequality (8) then becomes :

$$\int_{B(0,\delta)} {\varphi_\alpha}^{1+\epsilon_\alpha} dv_{g_\alpha} \leq C \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g}$$

Moreover,

$$\int_{B(0,\delta)} {\varphi_\alpha}^{1+\epsilon_\alpha} dv_{g_\alpha} \to C > 0 \tag{9}$$

by (7) and since $g_\alpha \to \xi$ in $C^1(B)$ for every ball B in $\mathbb{R}^n$. Finally, we get :

$$\frac{\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \geq C > 0$$

This ends the proof of step 1. Note that coming back to (8) and (9), one easily gets that :

$$\lim_{\alpha \to 0} {A_\alpha}^{\frac{n}{4}} \| u_\alpha \|_\infty = C > 0 \tag{10}$$

Step 2 *We recall that*

$$a_\alpha = A_\alpha^{\frac{1}{2}} = \left(\int_M u_\alpha^{1+\epsilon_\alpha} dv_g \right)^{\frac{2}{n(1+\epsilon_\alpha)}}$$

Let $(c_\alpha)_\alpha$ be a sequence of positive numbers such that : $\frac{a_\alpha}{c_\alpha} \to 0$ as $\alpha \to 0$. Then :

$$\lim_{\alpha \to 0} \frac{\int_{B_{x_\alpha}(c_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} = 1$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that :

$$\begin{array}{rc} (i) & \eta([0, \frac{1}{2}]) = \{1\} \\ (ii) & \eta([1, +\infty[) = \{0\} \\ (iii) & 0 \leq \eta \leq 1 \end{array}$$

For $k \in \mathbb{N}$, we let : $\eta_{\alpha,k}(x) = \big(\eta({c_\alpha}^{-1} d_g(x, x_\alpha))\big)^{2^k}$ where d_g denotes the distance for g. Multiplying (E_α) by ${\eta_{\alpha,k}}^2 u_\alpha$ and integrating over M gives :

$$2A_\alpha \int_M | \nabla \eta_{\alpha,k} u_\alpha |_g^2 dv_g - 2A_\alpha \int_M | \nabla \eta_{\alpha,k} |_g^2 {u_\alpha}^2 dv_g + \frac{4}{n} B_\alpha \int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g$$

$$= k_\alpha \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \tag{11}$$

Using $N(A_0(n) + \epsilon, B_\epsilon)(\eta_{\alpha,k} u_\alpha)$, one easily checks :

$$2A_\alpha \int_M | \nabla \eta_{\alpha,k} u_\alpha |_g^2 dv_g - 2A_\alpha \int_M | \nabla \eta_{\alpha,k} |_g^2 {u_\alpha}^2 dv_g + \frac{4}{n} B_\alpha \int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g$$

$$\begin{aligned} \leq k_\alpha \Big((A_0(n) + \epsilon) \int_M | \nabla \eta_{\alpha,k} u_\alpha |_g^2 dv_g (\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g)^{\frac{4}{n(1+\epsilon_\alpha)}} + \\ B_\epsilon \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g (\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g)^{\frac{4}{n(1+\epsilon_\alpha)}} \Big)^{\frac{n}{n+2}} \end{aligned} \tag{12}$$

Moreover, with the assumption on $(c_\alpha)_\alpha$:

$$| \nabla \eta_{\alpha,k} |_g^2 \leq \frac{C}{{c_\alpha}^2} \Rightarrow \lim_{\alpha \to 0} A_\alpha \int_M | \nabla \eta_{\alpha,k} |_g^2 {u_\alpha}^2 dv_g = 0$$

Now, let :

$$\begin{aligned}
\lambda_k &= \lim_{\alpha \to 0} \frac{\int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \\
\tilde{\lambda}_k &= \lim_{\alpha \to 0} \frac{\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g}
\end{aligned}$$

From the definition of $\eta_{\alpha,k}$, we get, for all $k \in \mathbb{N}$:

$$\lambda_{k+1} \leq \tilde{\lambda}_{k+1} \leq \lambda_k \leq \tilde{\lambda}_k \leq \mu = \lim_{\alpha \to 0} \frac{\int_{B_{x_\alpha}(c_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \tag{13}$$

and, by step 1 :

$$\exists C > 0 \text{ s.t. } \forall k \in \mathbb{N}, \lambda_k \geq C \tag{14}$$

Let us now prove that : $\lambda_k \leq \tilde{\lambda}_k^2$. Let $L_k = \lim_{\alpha \to 0} A_\alpha \int_M | \nabla \eta_{\alpha,k} u_\alpha |_g^2 dv_g$. Note that (4) and (5) imply :

$$\lim_{\alpha \to 0} B_\alpha \int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g = \lambda_k A_0(n)^{-1}$$

and

$$k_\alpha \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \leq C$$

In particular, (11) gives : $L_k < +\infty$. We also clearly have by (3) and (4) :

$$\lim_{\alpha \to 0} \int_M | \nabla \eta_{\alpha,k} u_\alpha |_g^2 dv_g \left(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g \right)^{\frac{4}{n(1+\epsilon_\alpha)}} = L_k \tilde{\lambda}_k^{\frac{4}{n}}$$

Equation (12) then leads to :

$$2L_k + \frac{4}{n} A_0(n)^{-1} \lambda_k \leq (2 + \frac{4}{n}) A_0(n)^{-1} ((A_0(n) + \epsilon) L_k \tilde{\lambda}_k^{\frac{4}{n}})^{\frac{n}{n+2}}$$

If $\tilde{L}_k = A_0(n) L_k$, we obtain, since ϵ was arbitrary :

$$2\tilde{L}_k + \frac{4}{n} \lambda_k \leq (2 + \frac{4}{n}) \tilde{L}_k^{\frac{n}{n+2}} \tilde{\lambda}_k^{\frac{4}{n+2}}$$

Let now, for x, y, z : $f(x,y,z) = (2+\frac{4}{n}) x^{\frac{n}{n+2}} y^{\frac{4}{n+2}} - (\frac{4}{n} z + 2x)$. Differentiating in x, we see that $\forall x, y, z > 0, f(x,y,z) \leq f(y^2, y, z)$, and then : $f(\tilde{L}_k, \tilde{\lambda}_k, \lambda_k) \leq f(\tilde{\lambda}_k^2, \tilde{\lambda}_k, \lambda_k) = \frac{4}{n}(\tilde{\lambda}_k^2 - \lambda_k)$. We then get : $\lambda_k \leq \tilde{\lambda}_k^2$. Now, from (13), (14), we get : $\forall N \in \mathbb{N},\ 0 < C \leq {\lambda_0}^N \leq \mu$. Since

$\mu \leq 1$, we have $\mu = 1$ which proves step 2. Note that we have also proved that $\tilde{L}_k = 1$ for all k. As one can check, we have then :

$$\lim_{\alpha \to 0} \frac{\int_{B_{x_\alpha}(c_\alpha)} |\nabla u_\alpha|_g^2 dv_g}{\int_M |\nabla u_\alpha|_g^2 dv_g} = 1 \tag{15}$$

As a consequence, we easily get from (11) :

$$\lim_{\alpha \to 0} \int_{B_{x_\alpha}(c_\alpha)} u_\alpha^2 dv_g = \lim_{\alpha \to 0} \frac{\int_{B_{x_\alpha}(c_\alpha)} u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 1 \tag{16}$$

Step 3 *There exists $C > 0$ such that, for all $x \in M$:*

$$u_\alpha(x) d(x, x_\alpha)^{\frac{n}{2}} \leq C$$

where d denotes the distance for g.

We proceed by contradiction. We suppose that the following assumption is true :

$$\exists y_\alpha \in M \text{ s.t. } \lim_{\alpha \to 0} u_\alpha(y_\alpha) d(y_\alpha, x_\alpha)^{\frac{n}{2}} = +\infty \tag{H}$$

Let :

$$v_\alpha = u_\alpha(y_\alpha) d(y_\alpha, x_\alpha)^{\frac{n}{2}}$$

We can assume that :

$$v_\alpha = \| u_\alpha(.) d(., x_\alpha)^{\frac{n}{2}} \|_\infty$$

First, we prove that, if ν is small enough :

$$B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}}) \cap B_{x_\alpha}(a_\alpha {v_\alpha}^\nu) = \emptyset \tag{17}$$

It is here sufficient to show that $d(x_\alpha, y_\alpha) \geq u_\alpha(y_\alpha)^{-\frac{2}{n}} + a_\alpha {v_\alpha}^\nu$, or, equivalently that ${v_\alpha}^{\frac{2}{n}-\nu} \geq {v_\alpha}^{-\nu} + a_\alpha u_\alpha(y_\alpha)^{\frac{2}{n}}$. If $\nu < \frac{2}{n}$, from (H), we get that ${v_\alpha}^{\frac{2}{n}-\nu} \to +\infty$ and ${v_\alpha}^{-\nu} \to 0$ as $\alpha \to 0$. Hence, it still has to be proved that $a_\alpha u_\alpha(y_\alpha)^{\frac{2}{n}} \leq C$. We have $a_\alpha u_\alpha(y_\alpha)^{\frac{2}{n}} \leq a_\alpha \| u_\alpha \|_\infty^{\frac{2}{n}}$. Since $a_\alpha = {A_\alpha}^{\frac{1}{2}}$ and by (10), this gives : $a_\alpha \| u_\alpha \|_\infty^{\frac{2}{n}} \leq C$. Equation (17) then follows. We let now, for $x \in B(0,1)$:

$$\begin{aligned} h_\alpha(x) &= (exp_{y_\alpha})^* g(l_\alpha x) \\ \psi_\alpha(x) &= u_\alpha(y_\alpha)^{-1} u_\alpha(exp_{y_\alpha}(l_\alpha x)) \end{aligned}$$

where :

$$l_\alpha = \| u_\alpha \|_\infty^{-\frac{n+4}{2n}} u_\alpha(y_\alpha)^{\frac{1}{2}}$$

On $B(0,1)$, we have :

$$\Delta_{h_\alpha} \psi_\alpha = \frac{k_\alpha \| u_\alpha \|_\infty^{-(1+\frac{4}{n})} u_\alpha(y_\alpha)}{2A_\alpha} \psi_\alpha - \frac{2B_\alpha \| u_\alpha \|_\infty^{-(1+\frac{4}{n})} u_\alpha(y_\alpha)^{\epsilon_\alpha}}{nA_\alpha} {\psi_\alpha}^{\epsilon_\alpha} \tag{E_α'}$$

Moreover :

$$h_\alpha \to \xi \text{ in } C^1(B(0,1)) \text{ as } \alpha \to 0 \tag{18}$$

We have $\| u_\alpha \|_{L^\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}}))} \leq C.u_\alpha(y_\alpha)$. To see this, note that, by the definition of y_α, we have for all $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})$:

$$u_\alpha(y_\alpha) d(x_\alpha, y_\alpha)^{\frac{n}{2}} \geq u_\alpha(x) d(x_\alpha, x)^{\frac{n}{2}} \tag{19}$$

Moreover, since $x \in B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})$:

$$d(y_\alpha, x) \leq u_\alpha(y_\alpha)^{-\frac{2}{n}}$$

and, by (H) : $u_\alpha(y_\alpha)^{-\frac{2}{n}} \leq \frac{1}{2} d(x_\alpha, y_\alpha)$. So we have :

$$d(x, x_\alpha) \geq d(x_\alpha, y_\alpha) - d(x, y_\alpha) \geq d(x_\alpha, y_\alpha) - u_\alpha(y_\alpha)^{-\frac{2}{n}} \geq \frac{1}{2} d(x_\alpha, y_\alpha)$$

Coming back to (19), the result follows immediately. Since $l_\alpha \leq u_\alpha(y_\alpha)^{-\frac{2}{n}}$, we then have $\| \psi_\alpha \|_{L^\infty(B(0,1))} \leq C$. From (6), (10) and the fact that, by (4), $B_\alpha {A_\alpha}^{\frac{n}{4}(1+\epsilon_\alpha)} \to C > 0$ as $\alpha \to 0$, we get

$$\lim_{\alpha \to 0} \| u_\alpha \|_\infty^{\epsilon_\alpha} = C \tag{20}$$

Now, from (6), (10) and (20), we see that (E'_α) has bounded coefficients and then :

$$\| \Delta_{h_\alpha} \psi_\alpha \|_{L^\infty(B(0,1))} \leq C$$

As in step 1, we get the existence of $\psi \in C^0(B(0,1))$ such that, up to a subsequence :

$$\psi_\alpha \to \psi \text{ in } C^0(B(0,1)) \text{ as } \alpha \to 0$$

Here, ψ is such that $\psi(0) = 1$ and then :

$$\int_{B(0,1)} \psi dx > 0 \tag{21}$$

However, by (18) :

$$\int_{B(0,1)} \psi dx = \lim_{\alpha \to 0} \int_{B(0,1)} {\psi_\alpha}^{1+\epsilon_\alpha} dv_{h_\alpha}$$

and, as one can check :

$$\int_{B(0,1)} {\psi_\alpha}^{1+\epsilon_\alpha} dv_{h_\alpha} = \beta_\alpha$$

where

$$\beta_\alpha = {A_\alpha}^{\frac{n}{4}(1+\epsilon_\alpha)} u_\alpha(y_\alpha)^{-(1+\epsilon_\alpha)} {l_\alpha}^{-n} \left(\frac{\int_{B_{y_\alpha}(l_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{{A_\alpha}^{\frac{n}{4}(1+\epsilon_\alpha)}} \right)$$

If we prove that $\lim_{\alpha \to 0} \beta_\alpha = 0$, we get a contradiction with (21) which ends the proof of step 3. First, let

$$m_\alpha = \frac{u_\alpha(y_\alpha)}{\| u_\alpha \|_\infty}$$

Clearly, by (10) :

$$\beta_\alpha \leq C m_\alpha^{-(\frac{n}{2}+1)} \left(\frac{\int_{B_{y_\alpha}(u_\alpha(l_\alpha))} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g} \right)$$

By step 2 and (17),

$$\lim_{\alpha\to 0}\left(\frac{\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})} {u_\alpha}^{1+\epsilon_\alpha} dv_g}{\int_M {u_\alpha}^{1+\epsilon_\alpha} dv_g}\right)=0 \tag{22}$$

If $m_\alpha \geq C > 0$, we have $\beta_\alpha \to 0$ as $\alpha \to 0$. Hence, we assume that $\lim_{\alpha\to 0} m_\alpha = 0$. We now proceed by induction to prove that :

$$\lim_{\alpha\to 0} m_\alpha^{-\left(\frac{n+3}{n+2}\right)^k} \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{2}{n}})} {u_\alpha}^2 dv_g = 0 \tag{H_k}$$

First, we prove that (H_0) is true. We proved before that

$$\| u_\alpha \|_{L^\infty(B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}}))} \leq C.u_\alpha(y_\alpha)$$

Hence, we have, noting that $u_\alpha(y_\alpha) \to \infty$ as $\alpha \to 0$:

$$\int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^2 dv_g \leq C u_\alpha(y_\alpha) \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^{1+\epsilon_\alpha} dv_g$$

$$\leq C m_\alpha \| u_\alpha \|_\infty \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^{1+\epsilon_\alpha} dv_g$$

By (10) and (22)

$$\lim_{\alpha\to 0} \| u_\alpha \|_\infty \int_{B_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^{1+\epsilon_\alpha} dv_g = 0$$

(H_0) then follows. Let now $\epsilon_k = \left(\frac{n+3}{n+2}\right)^k$ and suppose that (H_k) is true. Let us prove that (H_{k+1}) is true. Let $\eta_{\alpha,k}(x) = \eta(u_\alpha(y_\alpha)^{\frac{2}{n}} 2^k d_g(x, y_\alpha))$ where η is defined as in step 2. Multiplying (E_α) by

$$\frac{u_\alpha(\eta_{\alpha,k})^2}{m_\alpha^{\epsilon_k}}$$

and integrating over M, we obtain :

$$2A_\alpha m_\alpha^{-\epsilon_k} \int_M |\nabla \eta_{\alpha,k} u_\alpha|_g^2 dv_g - 2A_\alpha m_\alpha^{-\epsilon_k} \int_M |\nabla \eta_{\alpha,k}|_g^2 {u_\alpha}^2 dv_g$$

$$+\frac{4}{n} B_\alpha {m_\alpha}^{-\epsilon_k} \int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g = k_\alpha {m_\alpha}^{-\epsilon_k} \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \tag{23}$$

By (H_k) :

$$2A_\alpha m_\alpha^{-\epsilon_k} \int_M |\nabla \eta_{\alpha,k}|_g^2 {u_\alpha}^2 dv_g$$

$$\leq C A_\alpha u_\alpha(y_\alpha)^{\frac{4}{n}} m_\alpha^{-\epsilon_k} \int_{B_{y_\alpha}(2^{-k}u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^2 dv_g \leq C A_\alpha u_\alpha(y_\alpha)^{\frac{4}{n}}$$

Moreover, by (10), $A_\alpha u_\alpha(y_\alpha)^{\frac{4}{n}} = A_\alpha m_\alpha^{\frac{4}{n}} \| u_\alpha \|_\infty^{\frac{4}{n}} \leq C.m_\alpha^{\frac{4}{n}} \to 0$ as $\alpha \to 0$. We have also, by (H_k) and (5) :

$$\lim_{\alpha\to 0} k_\alpha {m_\alpha}^{-\epsilon_k} \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g = 0$$

Therefore, (23) gives :

$$2A_\alpha \int_M \mid \nabla \eta_{\alpha,k} u_\alpha \mid_g^2 dv_g \leq C.m_\alpha^{\epsilon_k} \tag{24}$$

$$\frac{4}{n} B_\alpha \int_M {\eta_{\alpha,k}}^2 {u_\alpha}^{1+\epsilon_\alpha} dv_g \leq C.m_\alpha^{\epsilon_k}$$

Up to replacing $\eta_{\alpha,k}$ by $\sqrt{\eta_{\alpha,k}}$, with the same arguments, we also have :

$$\frac{4}{n} B_\alpha \int_M {\eta_{\alpha,k}}^{1+\epsilon_\alpha} {u_\alpha}^{1+\epsilon_\alpha} dv_g \leq C.m_\alpha^{\epsilon_k} \tag{25}$$

Moreover, using $N(A,B)(\eta_{\alpha,k} u_\alpha)$, one easily checks that :

$$\left(\int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \right)^{\frac{n+2}{n}} \leq A. \int_M \mid \nabla \eta_{\alpha,k} u_\alpha \mid_g^2 dv_g \left(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g \right)^{\frac{4}{n(1+\epsilon_\alpha)}}$$

$$+B. \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \left(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g \right)^{\frac{4}{n(1+\epsilon_\alpha)}}$$

Clearly, we have in fact that :

$$\left(\int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \right)^{\frac{n+2}{n}} \leq C. \int_M \mid \nabla \eta_{\alpha,k} u_\alpha \mid_g^2 dv_g \left(\int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g \right)^{\frac{4}{n(1+\epsilon_\alpha)}}$$

$$\leq \frac{C}{A_\alpha B_\alpha^{\frac{4}{n(1+\epsilon_\alpha)}}} \left(\int_M \mid \nabla \eta_{\alpha,k} u_\alpha \mid_g^2 dv_g A_\alpha \right) \left(B_\alpha \int_M (\eta_{\alpha,k} u_\alpha)^{1+\epsilon_\alpha} dv_g \right)^{\frac{4}{n(1+\epsilon_\alpha)}}$$

Using (24) and (25), we get

$$\left(\int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g \right)^{\frac{n+2}{n}} \leq \frac{C}{A_\alpha B_\alpha^{\frac{4}{n(1+\epsilon_\alpha)}}} .m_\alpha^{(1+\frac{4}{n(1+\epsilon_\alpha)})\epsilon_k}$$

By (4), $A_\alpha B_\alpha^{\frac{4}{n(1+\epsilon_\alpha)}} \geq C > 0$. Since :

$$\int_{B_{y_\alpha}(2^{-(k+1)} u_\alpha(y_\alpha)^{-\frac{2}{n}})} u_\alpha^2 dv_g \leq \int_M (\eta_{\alpha,k} u_\alpha)^2 dv_g$$

(H_{k+1}) then follows. As a consequence, (H_k) is true for all k. Coming back to (25) , we get that, for all k :

$$\lim_{\alpha \to 0} m_\alpha^{-\epsilon_k} B_\alpha \int_{B_{y_\alpha}(2^{-k} u_\alpha(y_\alpha)^{-\frac{2}{n}})} {u_\alpha}^{1+\epsilon_\alpha} dv_g = 0$$

Using the fact that $\lim_{\alpha \to 0} l_\alpha u_\alpha(y_\alpha)^{\frac{2}{n}} = 0$ and choosing k such that $\epsilon_k \geq \frac{n}{2} + 1$, we get : $\lim_{\alpha \to 0} \beta_\alpha = 0$ which ends the proof of step 3.

Step 4 *For all $c, k > 0$, we have :*

$$\lim_{\alpha \to 0} {A_\alpha}^{-k} \int_{M - B_{x_\alpha}(c)} {u_\alpha}^2 dv_g \quad = \quad 0 \tag{26}$$

$$\lim_{\alpha\to 0} A_\alpha{}^{-k}\int_{M-B_{x_\alpha}(c)} |\nabla u_\alpha|_g^2 dv_g \quad = \quad 0 \tag{27}$$

$$\lim_{\alpha\to 0} A_\alpha{}^{-k}\int_{M-B_{x_\alpha}(c)} u_\alpha{}^{1+\epsilon_\alpha} dv_g \quad = \quad 0 \tag{28}$$

Let $r_\alpha(x) = d_g(x, x_\alpha)$ and let $\delta \in]0, \frac{n}{4}[$. Using step 3, we have :

$$A_\alpha{}^{-\delta}\int_{M-B_{x_\alpha}(c)} u_\alpha{}^2 dv_g \leq C.A_\alpha{}^{-\delta}\int_{M-B_{x_\alpha}(c)} u_\alpha{}^{1+\epsilon_\alpha} r_\alpha{}^{-\frac{n}{2}(1-\epsilon_\alpha)} dv_g$$

$$\leq C.A_\alpha{}^{-\delta}\int_{M-B_{x_\alpha}(c)} u_\alpha{}^{1+\epsilon_\alpha} dv_g$$

Recall the definition of A_α to get :

$$\lim_{\alpha\to 0} A_\alpha{}^{-\delta}\int_{M-B_{x_\alpha}(c)} u_\alpha{}^2 dv_g = 0$$

Mimicking what we have done in the proof of step 3, we prove by induction that, for all k :

$$\lim_{\alpha\to 0} A_\alpha{}^{-\left(\frac{n+3}{n+2}\right)^k\delta}\int_{M-B_{x_\alpha}(2^k c)} u_\alpha{}^2 dv_g = 0$$

This gives (26). Following the arguments used in the proof of step 3, one easily gets (27) and (28) from (24) and (25). Now, we set, for $c > 0$ small, $\eta_\alpha = \eta(c^{-1}r_\alpha)$ where η is as above. We also define :

$$\begin{aligned} r_\nabla &= \frac{\int_M |\nabla u_\alpha \eta_\alpha|_g^2 R_{ij}(x_\alpha)x^i x^j dv_g}{\int_M |\nabla u_\alpha \eta_\alpha|_g^2 dv_g} \\ r_1 &= \frac{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} R_{ij}(x_\alpha)x^i x^j dv_g}{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} dv_g} \\ r_2 &= \frac{\int_M (u_\alpha \eta_\alpha)^2 R_{ij}(x_\alpha)x^i x^j dv_g}{\int_M (u_\alpha \eta_\alpha)^2 dv_g} \end{aligned}$$

where $(x^1, .., x^n)$ are exponential coordinates.

Step 5 *We have*

$$\lim_{\alpha\to 0}\frac{-\frac{1}{6}\left(-r_\nabla + (1+\frac{2}{n})r_2 - \frac{4}{n(1+\epsilon_\alpha)}r_1\right)}{A_\alpha}$$

$$= \frac{|\mathcal{B}|^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2} + \frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}} S_g(x_0) \tag{29}$$

We come back to the notations of step 1. Let :

$$C_0 = \lim_{\alpha\to 0} \| u_\alpha \|_\infty^{-1} A_\alpha^{-\frac{n}{4}} \text{ and } \tilde{C}_0 = \lim_{\alpha\to 0} A_\alpha^{\epsilon_\alpha}$$

Note that, by (10) and (20), these limits exist. As one easily checks :

$$\int_{B(0,\delta)} \varphi_\alpha{}^2 dv_{g_\alpha} = \| u_\alpha \|_\infty{}^{-2} A_\alpha{}^{-\frac{n}{2}}\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha{}^2 dv_g$$

and

$$\int_{B(0,\delta)} {\varphi_\alpha}^{1+\epsilon_\alpha} dv_{g_\alpha} = \| u_\alpha \|_\infty^{-(1+\epsilon_\alpha)} {A_\alpha}^{-\frac{n}{2}} \int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g$$

$$= \left(\| u_\alpha \|_\infty^{-(1+\epsilon_\alpha)} {A_\alpha}^{-\frac{n}{4}(1+\epsilon_\alpha)}\right) \left({A_\alpha}^{-\frac{n}{4}(1+\epsilon_\alpha)} \int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} dv_g\right) {A_\alpha}^{\frac{n}{2}\epsilon_\alpha}$$

Let first α goes to 0 and then, δ to $+\infty$. By (16) and step 2, we have :

$$\int_{\mathbb{R}^n} \varphi^2 dv_\xi = {C_0}^2 \tag{30}$$

and

$$\int_{\mathbb{R}^n} \varphi dv_\xi = C_0 \tilde{C}_0^{\frac{n}{2}} \tag{31}$$

Now, let us compute $\int_{\mathbb{R}^n} | \nabla\varphi |_\xi^2 dv_\xi$. First, it is clear that :

$$\varphi_\alpha \to \varphi \text{ in } C^1(B) \text{ as } \alpha \to 0 \tag{32}$$

for all compact ball B in $\mathbb{R}^n$. Let $\eta_\delta(x) = \eta\left((2\delta)^{-1} \mid x \mid\right)$ where η is as in step 2. Multiply $(\tilde{E}_\alpha)$ by $\varphi_\alpha {\eta_\delta}^2$ and integrate over $\mathbb{R}^n$. We check :

$$\int_{\mathbb{R}^n} < \nabla\varphi_\alpha, \nabla\varphi_\alpha \eta_\delta^2 >_{g_\alpha} dv_{g_\alpha} + \frac{2B_\alpha}{n\| u_\alpha \|_\infty^{1-\epsilon_\alpha}} \int_{\mathbb{R}^n} {\varphi_\alpha}^{1+\epsilon_\alpha} {\eta_\delta}^2 dv_{g_\alpha} = \frac{k_\alpha}{2} \int_{\mathbb{R}^n} {\varphi_\alpha}^2 {\eta_\delta}^2 dv_{g_\alpha}$$

Using (4), one easily gets :

$$\lim_{\alpha\to 0} \frac{2B_\alpha}{n\| u_\alpha \|_\infty^{1-\epsilon_\alpha}} = \frac{2}{n} A_0(n)^{-1} C_0 \tilde{C}_0^{-\frac{n}{2}}$$

and then, by (5) and (32) :

$$\int_{\mathbb{R}^n} < \nabla\varphi, \nabla\varphi\eta_\delta^2 >_\xi dv_\xi + \frac{2}{n} A_0(n)^{-1} C_0 \tilde{C}_0^{-\frac{n}{2}} \int_{\mathbb{R}^n} {\eta_\delta}^2 \varphi dv_\xi$$

$$= (1 + \frac{2}{n}) A_0(n)^{-1} \int_{\mathbb{R}^n} {\eta_\delta}^2 \varphi^2 dv_\xi \tag{33}$$

We have

$$\int_{\mathbb{R}^n} < \nabla\varphi, \nabla\varphi\eta_\delta^2 >_\xi dv_\xi = 2\int_{\mathbb{R}^n} < \nabla\varphi, \nabla\eta_\delta >_\xi \varphi\eta_\delta dv_\xi + \int_{\mathbb{R}^n} | \nabla\varphi |_\xi^2 {\eta_\delta}^2 dv_\xi$$

$$\leq 2\left(\int_{\mathbb{R}^n} | \nabla\eta_\delta |_\xi^2 \varphi^2 dv_\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} | \nabla\varphi |_\xi^2 {\eta_\delta}^2 dv_\xi\right)^{\frac{1}{2}} + \int_{\mathbb{R}^n} | \nabla\varphi |_\xi^2 {\eta_\delta}^2 dv_\xi$$

By (30) and since $| \nabla\eta_\delta | \leq \frac{\text{cst}}{\delta}$, one easily gets :

$$\lim_{\delta\to+\infty} \int_{\mathbb{R}^n} < \nabla\varphi, \nabla\varphi\eta_\delta^2 >_\xi dv_\xi = \int_{\mathbb{R}^n} | \nabla\varphi |_\xi^2 dv_\xi \tag{34}$$

By (30), we know that $\varphi \in L^2(\mathbb{R}^n)$. As a consequence, plugging (34) into (33) and using (31), we have :

$$\int_{\mathbb{R}^n} \mid \nabla \varphi \mid_\xi^2 dv_\xi = A_0(n)^{-1} {C_0}^2 \tag{35}$$

Now, let, for $u \in H_1^2(\mathbb{R}^n)$:

$$I_\xi(u) = \frac{\int_{\mathbb{R}^n} \mid \nabla u \mid_\xi^2 dv_\xi (\int_{\mathbb{R}^n} u dv_\xi)^{\frac{4}{n}}}{(\int_{\mathbb{R}^n} u^2 dv_\xi)^{1+\frac{2}{n}}}$$

By the works of Carlen and Loss [3], we know that :

$$\forall u \in H_1^2(\mathbb{R}^n), I_\xi(u) \geq A_0(n)^{-1}$$

By (30), (31) and (35), we have :

$$I_\xi(\varphi) = A_0(n)^{-1} \tilde{C}_0^2$$

Since $\tilde{C}_0 \leq 1$, it follows that $\tilde{C}_0 = 1$ (if $\tilde{C}_0 < 1$, we would have $I_\xi(\varphi) < A_0(n)^{-1}$). Therefore, $I_\xi(\varphi) = A_0(n)^{-1}$. Let u, $u \not\equiv 0$ and radially symetric, be an eigenfunction associated to λ_1, the first eigenvalue of the Laplacian on the unit ball $\mathcal{B}$ in $\mathbb{R}^n$ for radial functions with Neumann condition on the boundary. Moreover, we may assume that u(0)=1. By Carlen and Loss [3], we have :

$$\varphi = kv(\lambda x)$$

where $v(x) = u(x) - u(1)$. Now, by (30), (31) and since $\tilde{C}_0 = 1$, we get :

$$\int_{\mathbb{R}^n} \varphi^2 dv_\xi = \left(\int_{\mathbb{R}^n} \varphi dv_g \right)^2$$

We know that (see theorem 1.3 in [6]) :

$$\int_{\mathbb{R}^n} v^2 dv_\xi = \frac{n+2}{2} u(1)^2 \mid \mathcal{B} \mid$$

$$\int_{\mathbb{R}^n} v dv_\xi = - \mid \mathcal{B} \mid u(1)$$

This gives then :

$$\lambda^2 = \lambda_0^2$$

where

$$\lambda_0^2 = \left(\frac{n+2}{2} \right)^{-\frac{2}{n}} \mid \mathcal{B} \mid^{\frac{2}{n}}$$

Let now :

$$\begin{aligned}
r_{\nabla,\delta} &= \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \mid_g^2 R_{ij}(x_\alpha) x^i x^j dv_g}{\int_M \mid \nabla u_\alpha \eta_\alpha \mid^2 dv_g} \\
r_{1,\delta} &= \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^{1+\epsilon_\alpha} R_{ij}(x_\alpha) x^i x^j dv_g}{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} dv_g} \\
r_{2,\delta} &= \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^2 R_{ij}(x_\alpha) x^i x^j dv_g}{\int_M (u_\alpha \eta_\alpha)^2 dv_g}
\end{aligned}$$

We recall that $\eta_\alpha = \eta(c^{-1}r_\alpha)$ where $c > 0$ is small and where η is defined as before. Using (15), we easily see that

$$\lim_{\alpha\to 0} \frac{\int_M \mid \nabla u_\alpha \eta_\alpha \mid_g^2 dv_g}{\int_M \mid \nabla u_\alpha \mid_g^2 dv_g} = 1$$

We also get that, with step 2 and (16),

$$\lim_{\alpha\to 0} \frac{\int_M (u_\alpha \eta_\alpha)^2 dv_g}{\int_M u_\alpha^2 dv_g} = 1$$

$$\lim_{\alpha\to 0} \frac{\int_M (u_\alpha \eta_\alpha)^{1+\epsilon_\alpha} dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha} dv_g} = 1$$

Now, by an easy proof by contradiction using step 2, (15) and (16), we see that

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_M \mid \nabla u_\alpha \mid_g^2 dv_g}{\int_{B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \mid_g^2 dv_g} = 1$$

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_M u_\alpha^2 dv_g}{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^2 dv_g} = 1$$

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_M u_\alpha^{1+\epsilon_\alpha} dv_g}{\int_{B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha} dv_g} = 1$$

Here, $\lim_{\delta\to\infty} \lim_{\alpha\to 0}$ means that α first goes to 0 and then, δ goes to $+\infty$. This implies that :

$$\begin{aligned}
\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{r_{\nabla,\delta}}{A_\alpha} &= \lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \mid_g^2 R_{ij}(x_\alpha) x^i x^j dv_g}{A_\alpha \int_{B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \mid_g^2 dv_g} \\
\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{r_{1,\delta}}{A_\alpha} &= \lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^{1+\epsilon_\alpha} R_{ij}(x_\alpha) x^i x^j dv_g}{A_\alpha \int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^{1+\epsilon_\alpha} dv_g} \\
\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{r_{2,\delta}}{A_\alpha} &= \lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{\int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^2 R_{ij}(x_\alpha) x^i x^j dv_g}{A_\alpha \int_{B_{x_\alpha}(\delta a_\alpha)} (u_\alpha)^2 dv_g}
\end{aligned}$$

Let $(y^1, .., y^n)$ be canonical coordinates in $\mathbb{R}^n$ and $(x^1, .., x^n)$ be exponential coordinates in M. It is easy to see that, for a radial function f:

$$\int_{B(0,\delta)} f y^i y^j dv_\xi = \delta^{ij} \frac{1}{n} \int_{B(0,\delta)} f \mid y \mid^2 dv_\xi$$

We also have :

$$\int_{B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^p x^i x^j dv_g = \parallel u_\alpha \parallel_\infty^p {A_\alpha}^{1+\frac{n}{2}} \int_{B(0,\delta)} {\varphi_\alpha}^p y^i y^j dv_{g_\alpha}$$

and :

$$\int_{B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \mid_g^2 x^i x^j dv_g = \parallel u_\alpha \parallel_\infty^2 {A_\alpha}^{\frac{n}{2}} \int_{B(0,\delta)} \mid \nabla \varphi_\alpha \mid_{g_\alpha} y^i y^j dv_{g_\alpha}$$

By these results and noting that φ is compactly supported, we have, for δ large enough :

$$\lim_{\alpha\to 0} \frac{r_{\nabla,\delta}}{A_\alpha} = \frac{S_g(x_0)}{n} \frac{\int_{\mathbb{R}^n} \mid \nabla \varphi \mid_\xi^2 \mid y \mid^2 dv_\xi}{\int_{\mathbb{R}^n} \mid \nabla \varphi \mid_\xi^2 dv_\xi}$$

$$\lim_{\alpha\to 0}\frac{r_{1,\delta}}{A_\alpha}=\frac{S_g(x_0)}{n}\frac{\int_{\mathbb{R}^n}\varphi|\ y\ |^2 dv_\xi}{\int_{\mathbb{R}^n}\varphi dv_\xi}$$

$$\lim_{\alpha\to 0}\frac{r_{2,\delta}}{A_\alpha}=\frac{S_g(x_0)}{n}\frac{\int_{\mathbb{R}^n}\varphi^2|\ y\ |^2 dv_\xi}{\int_{\mathbb{R}^n}\varphi^2 dv_\xi}$$

Then, for $\delta\geq\lambda_0$:

$$\lim_{\alpha\to 0}\frac{-\frac{1}{6}\left(-r_{\nabla,\delta}+(1+\frac{2}{n})r_{2,\delta}-\frac{4}{n(1+\epsilon_\alpha)}r_{1,\delta}\right)}{A_\alpha}=\frac{{\lambda_0}^{-2}S_g(x_0)}{6n}\Big(-\frac{\int_{\mathbb{R}^n}|\ \nabla v\ |_\xi^2|\ y\ |^2 dv_\xi}{\int_{\mathbb{R}^n}|\ \nabla v\ |_\xi^2 dv_\xi}$$

$$+\frac{n+2}{n}\frac{\int_{\mathbb{R}^n}v^2|\ y\ |^2 dv_\xi}{\int_{\mathbb{R}^n}v^2 dv_\xi}-\frac{4}{n(1+\epsilon_\alpha)}\frac{\int_{\mathbb{R}^n}v|\ y\ |^2 dv_\xi}{\int_{\mathbb{R}^n}v dv_\xi}\Big)$$

This expression has been computed in Druet, Hebey and Vaugon [6]. We have :

$$\frac{-\frac{1}{6}\left(-r_{\nabla,\delta}+(1+\frac{2}{n})r_{2,\delta}-\frac{4}{n(1+\epsilon_\alpha)}r_{1,\delta}\right)}{A_\alpha}=\frac{|\ \mathcal{B}\ |^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2}+\frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}}S_g(x_0)$$

Hence, it is sufficient to prove that :

$$\lim_{\delta\to\infty}\lim_{\alpha\to 0}\frac{r_{\nabla,\delta}-r_\nabla}{A_\alpha}=\lim_{\delta\to\infty}\lim_{\alpha\to 0}\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}|\ \nabla u_\alpha\eta_\alpha\ |_g^2 R_{ij}(x_\alpha)x^ix^j dv_g}{A_\alpha\int_M|\ \nabla u_\alpha\eta_\alpha\ |_g^2 dv_g}=0 \tag{36}$$

$$\lim_{\delta\to\infty}\lim_{\alpha\to 0}\frac{r_{1,\delta}-r_1}{A_\alpha}=\lim_{\delta\to\infty}\lim_{\alpha\to 0}\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}(u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}R_{ij}(x_\alpha)x^ix^j dv_g}{A_\alpha\int_M(u_\alpha\eta_\alpha)^{1+\epsilon_\alpha}dv_g}=0 \tag{37}$$

$$\lim_{\delta\to\infty}\lim_{\alpha\to \alpha_0}\frac{r_{2,\delta}-r_2}{A_\alpha}=\lim_{\delta\to\infty}\lim_{\alpha\to \alpha_0}\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}(u_\alpha\eta_\alpha)^2 R_{ij}(x_\alpha)x^ix^j dv_g}{A_\alpha\int_M(u_\alpha\eta_\alpha)^2 dv_g}=0 \tag{38}$$

First, let us deal with (38). Let :

$$T_\alpha=\left|\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}(\eta_\alpha u_\alpha)^2 R_{ij}(x_\alpha)x^ix^j dv_g}{A_\alpha\int_M(\eta_\alpha u_\alpha)^2 dv_g}\right|$$

By (16) :

$$T_\alpha\leq C\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}u_\alpha^2{r_\alpha}^2 dv_g}{A_\alpha}$$

Now, by step 3 :

$$T_\alpha\leq C\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}u_\alpha^{\epsilon_\alpha}{r_\alpha}^{2-n}{r_\alpha}^{\frac{n}{2}\epsilon_\alpha}dv_g}{A_\alpha}$$

$$\leq C\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)}u_\alpha^{\epsilon_\alpha}{r_\alpha}^{2-n}dv_g}{A_\alpha}\leq C\frac{{A_\alpha}^{1-\frac{n}{2}}\int_{M-B_{x_\alpha}(\delta a_\alpha)}u_\alpha^{\epsilon_\alpha}dv_g}{A_\alpha}$$

To estimate this expression, we integrate (E_α) over $M-B_{x_\alpha}(\delta a_\alpha)$. We get :

$$T_\alpha \leq C\left(\frac{{A_\alpha}^{-\frac{n}{2}}}{B_\alpha}\int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha dv_g + \frac{{A_\alpha}^{1-\frac{n}{2}}}{B_\alpha}\int_{\partial B_{x_\alpha}(\delta a_\alpha)} \partial_\nu u_\alpha d\sigma\right) \tag{39}$$

Let us prove that the second member of (39) goes to 0 if we let α goes to 0 and δ to ∞. We have, using the definition of A_α :

$$\frac{{A_\alpha}^{-\frac{n}{2}}}{B_\alpha}\int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha dv_g \leq \frac{{A_\alpha}^{-\frac{n}{4}}}{B_\alpha}\left(\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha} dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha} dv_g}\right)^{\frac{1}{1+\epsilon_\alpha}}$$

By (4), we have :

$$\lim_{\alpha\to 0}\frac{{A_\alpha}^{-\frac{n}{4}}}{B_\alpha} = C$$

Step 2 clearly implies that :

$$\lim_{\delta\to+\infty}\lim_{\alpha\to 0}\left(\frac{\int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha^{1+\epsilon_\alpha} dv_g}{\int_M u_\alpha^{1+\epsilon_\alpha} dv_g}\right)^{\frac{1}{1+\epsilon_\alpha}} = 0$$

Hence :

$$\lim_{\delta\to\infty}\lim_{\alpha\to 0}\frac{{A_\alpha}^{-\frac{n}{2}}}{B_\alpha}\int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha dv_g = 0$$

Now, if $r_\alpha = \delta a_\alpha$, we have :

$$\mid \partial_\nu u_\alpha(x) \mid \leq \frac{\parallel u_\alpha \parallel_\infty}{{A_\alpha}^{\frac{1}{2}}} \parallel (\nabla\varphi)_g \parallel_{L^\infty(\partial B(0,\delta))}$$

Since φ is compactly supported (see above), for δ large enough :

$$\parallel (\nabla\varphi_\alpha)_{g_\alpha} \parallel_{L^\infty(\partial B(0,\delta))} \to 0$$

Consequently, for δ large enough :

$$\lim_{\alpha\to 0}\frac{{A_\alpha}^{1-\frac{n}{2}}}{B_\alpha}\int_{\partial B_{x_\alpha}(\delta a_\alpha)} \partial_\nu u_\alpha d\sigma = 0$$

By (39), this proves (38). To get (36) and (37), multiply (E_α) by $\frac{{r_\alpha}^2{\eta_\alpha}^2 u_\alpha}{A_\alpha}$ and integrate over $M - B_{x_\alpha}(\delta a_\alpha)$:

$$-2\int_{\partial B_{x_\alpha}(\delta a_\alpha)} (\partial_\nu u_\alpha) u_\alpha {r_\alpha}^2{\eta_\alpha}^2 d\sigma + 2\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha r_\alpha \mid_g^2 dv_g - 2\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla\eta_\alpha r_\alpha \mid_g^2 {u_\alpha}^2 dv_g$$

$$+\frac{4B_\alpha}{nA_\alpha}\int_{M-B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha}{r_\alpha}^2{\eta_\alpha}^2 dv_g = \frac{k_\alpha}{A_\alpha}\int_{M-B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^2{r_\alpha}^2{\eta_\alpha}^2 dv_g \tag{40}$$

As we did before, we use the fact that for $r_\alpha = \delta a_\alpha$:

$$\mid \partial_\nu u_\alpha(x) \mid \leq \frac{\parallel u_\alpha \parallel_\infty}{{A_\alpha}^{\frac{1}{2}}} \parallel (\nabla\varphi_\alpha)_g \parallel_{L^\infty(\partial B(0,\delta))}$$

and :

$$u_\alpha(x) \leq \parallel u_\alpha \parallel_\infty \parallel \varphi_\alpha \parallel_{L^\infty(\partial B(0,\delta))}$$

This gives that for δ large enough, the boundary term goes to 0. Moreover, it is clear that we have :

$$\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla \eta_\alpha r_\alpha \mid_g^2 u_\alpha^2 dv_g \leq C \int_{M-B_{x_\alpha}(\delta a_\alpha)} u_\alpha^2 dv_g$$

By step 2, we obtain :

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla r_\alpha \eta_\alpha^2 \mid_g^2 {u_\alpha}^2 dv_g = 0$$

Observe that the second member of (40) goes to 0 when $\alpha \to 0$ and $\delta \to \infty$. This easily follows from what we did when we proved (38). Relation (40) then implies that :

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha r_\alpha \mid_g^2 dv_g = 0 \tag{41}$$

and also that :

$$\lim_{\delta\to\infty} \lim_{\alpha\to 0} \frac{4B_\alpha}{nA_\alpha} \int_{M-B_{x_\alpha}(\delta a_\alpha)} {u_\alpha}^{1+\epsilon_\alpha} {r_\alpha}^2 {\eta_\alpha}^2 dv_g = 0$$

which gives (37). In addition :

$$\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha r_\alpha \mid_g^2 dv_g = \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha \mid_g^2 {r_\alpha}^2 dv_g$$

$$+2 \int_{M-B_{x_\alpha}(\delta a_\alpha)} < \nabla u_\alpha \eta_\alpha, \nabla r_\alpha >_g u_\alpha \eta_\alpha r_\alpha dv_g + \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla r_\alpha \mid_g^2 \eta_\alpha {u_\alpha}^2 dv_g$$

For every $x, y, \epsilon > 0$, we have : $xy \leq \frac{1}{2}(\epsilon x^2 + \frac{1}{\epsilon} y^2)$. Noting that :

$$\int_{M-B_{x_\alpha}(\delta a_\alpha)} < \nabla u_\alpha \eta_\alpha, \nabla r_\alpha >_g u_\alpha \eta_\alpha r_\alpha dv_g$$

$$\geq -\left(\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha \mid_g^2 {r_\alpha}^2 dv_g \right)^{\frac{1}{2}} \left(\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla r_\alpha \mid_g^2 \eta_\alpha {u_\alpha}^2 dv_g \right)^{\frac{1}{2}}$$

we get :

$$\int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha r_\alpha \mid_g^2 dv_g \geq (1-\epsilon) \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla u_\alpha \eta_\alpha \mid_g^2 {r_\alpha}^2 dv_g$$

$$+(1-\frac{1}{\epsilon}) \int_{M-B_{x_\alpha}(\delta a_\alpha)} \mid \nabla r_\alpha \mid_g^2 (\eta_\alpha u_\alpha)^2 dv_g$$

Using (41) and the fact that $\lim A_\alpha \int_M \mid \nabla u_\alpha \eta_\alpha \mid_g^2 dv_g = A_0(n)^{-1}$, we then clearly get (36). Finally, this proves step 5.

Step 6 *We prove the theorem.*

Let, for $u \in H_1^2(M)$:

$$I_{g,\alpha}(u) = I_\alpha(u) - (\alpha_0 - \alpha)(\int_M \mid u \mid^{1+\epsilon_\alpha} dv_g)^{\frac{4}{n(1+\epsilon_\alpha)}}$$

a– *We first prove that :*

$$\lim_{\alpha\to 0} \frac{A_0(n)^{-1} - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} = \alpha_0 \tag{42}$$

By (26), (27) and (28), one can check that :

$$\lim_{\alpha\to 0} \frac{I_{g,\alpha}(u_\alpha) - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} = 0 \tag{43}$$

Moreover, we have :

$$I_{g,\alpha}(u_\alpha) = I_\alpha(u_\alpha) - (\alpha_0 - \alpha)A_\alpha$$

Since $\alpha \to 0$ and $I_\alpha(u_\alpha) \le A_0(n)^{-1}$, we get :

$$\liminf_{\alpha\to 0} \frac{A_0(n)^{-1} - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} \ge \alpha_0 \tag{44}$$

In addition, we can also write, by (43)

$$\limsup_{\alpha\to 0} \frac{A_0(n)^{-1} - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} = \limsup_{\alpha\to 0} \frac{A_0(n)^{-1} - I_0(u_\alpha) + \alpha_0 A_\alpha}{A_\alpha}$$

By definition of α_0, we have $I_0(u_\alpha) \ge \mu_0 = A_0(n)^{-1}$. This implies that :

$$\limsup_{\alpha\to 0} \frac{A_0(n)^{-1} - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} \le \alpha_0 \tag{45}$$

(42) then comes from (43), (44) and (45).

b$-$ *We prove that* :

$$\int_M |\nabla \eta_\alpha u_\alpha|_\xi^2 dv_\xi - \int_M |\nabla \eta_\alpha u_\alpha|_g^2 dv_g = -\frac{1}{6}\int_M |\nabla \eta_\alpha u_\alpha|_\xi^2 R_{ij}(x_\alpha)x^i x^j dv_g + O(1) \tag{46}$$

First note that the limit of right-hand side member of (46) exists. We have

$$\int_M |\nabla \eta_\alpha u_\alpha|_g^2 dv_g = \int_M |\nabla \eta_\alpha u_\alpha|_\xi^2 dv_g + \int_M (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha \eta_\alpha^2 dv_g + C_1(\alpha) \tag{47}$$

where $C_1(\alpha)$ stands for the terms in which the derivatives of η_α appear. Since $supp(\nabla\eta_\alpha) \subset M - B_{x_\alpha}(\frac{c}{2})$ and by step 2, (15) and (16), we see that $C_1(\alpha) \to 0$ when $\alpha \to 0$. We write that, for $\delta > 0$,

$$\left|\int_M (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha \eta_\alpha^2 dv_g\right| \le \left|\int_{B_{x_\alpha}(\delta a_\alpha)} (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha dv_g\right|$$

$$+\left|\int_{M-B_{x_\alpha}(\delta a_\alpha)} (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha \eta_\alpha^2 dv_g\right|$$

Using the Cartan Hadamard expansion of the metric g, we get that

$$\left|\int_M (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha \eta_\alpha^2 dv_g\right| \le C\left|\int_{B_{x_\alpha}(\delta a_\alpha)} R^i{}_{kl}{}^j(x_\alpha)\partial_i u_\alpha \partial_j u_\alpha x^k x^l dv_g\right|$$

$$+C\int_{B_{x_\alpha}(\delta a_\alpha)} |\nabla u_\alpha|_g^2 r_\alpha^3 dv_g + C\int_{M-B_{x_\alpha}(\delta a_\alpha)} |\nabla u_\alpha|_g^2 r_\alpha^2 dv_g$$

where $(R^i{}_{kl}{}^j(x_\alpha))$ are the components of the Riemann curvature of g in exponential map at x_α. One gets from (41) that the third term of this expression is small if δ is large. The second term goes to 0 when α tends to 0. It can be seen by writing that, on $B_{x_\alpha}(\delta a_\alpha)$, $r_\alpha \le \delta a_\alpha$. We now prove that the first term goes to 0 with α. We write that

$$\left| \int_{B_{x_\alpha}(\delta a_\alpha)} R^i{}_{kl}{}^j(x_\alpha)\partial_i u_\alpha \partial_j u_\alpha x^k x^l dv_g \right|$$

$$\le C \| u_\alpha \|_\infty^2 A_\alpha^{\frac{n}{2}} \left| \int_{B(0,\delta)} R^i{}_{kl}{}^j(x_\alpha)\partial_i \varphi_\alpha \partial_j \varphi_\alpha x^k x^l dv_{g_\alpha} \right|$$

where φ is defined as in step 1. Now, since $\varphi_\alpha \to \varphi$ in $C^1(B(0,\delta))$ when $\alpha \to 0$ and since φ is radially symmetric, we get that

$$\lim_{\alpha \to 0} R^i{}_{kl}{}^j(x_\alpha)\partial_i u_\alpha \partial_j u_\alpha x^k x^l = 0$$

Together with (10), this proves that, for all δ,

$$\lim_{\alpha \to 0} \int_{B_{x_\alpha}(\delta a_\alpha)} R^i{}_{kl}{}^j(x_\alpha)\partial_i \varphi_\alpha \partial_j \varphi_\alpha x^k x^l dv_g = 0$$

We finally obtain that

$$\lim_{\alpha \to 0} \int_M (g^{ij} - \delta^{ij})\partial_i u_\alpha \partial_j u_\alpha \eta_\alpha^2 dv_g = 0 \tag{48}$$

To conclude, we write that, by the Cartan Hadamard expansion of g,

$$\int_M | \nabla \eta_\alpha u_\alpha |_\xi^2 dv_g = \int_M | \nabla \eta_\alpha u_\alpha |_\xi^2 dv_\xi + \frac{1}{6}\int_M | \nabla \eta_\alpha u_\alpha |_\xi^2 R_{ij}(x_\alpha)x^i x^j dv_g + O(1) \tag{49}$$

We then get (46) from (47), (48) and (49).

c$-$ *We prove that :*

$$\lim_{\alpha \to 0} \frac{I_{\xi,\alpha}(\eta_\alpha u_\alpha) - I_{g,\alpha}(\eta_\alpha u_\alpha)}{A_\alpha} = A_0(n)^{-1} \frac{| \mathcal{B} |^{-\frac{2}{n}}}{6n} \left(\frac{2}{n+2} + \frac{n-2}{\lambda_1} \right) \left(\frac{n+2}{2} \right)^{\frac{2}{n}} S_g(x_0) \tag{50}$$

where I_ξ is defined as above.
Let :

$$\begin{aligned} t_1 &= \frac{\int_M (\eta_\alpha u_\alpha)^{1+\epsilon_\alpha} dv_\xi - \int_M (\eta_\alpha u_\alpha)^{1+\epsilon_\alpha} dv_g}{\int_M (\eta_\alpha u_\alpha)^{1+\epsilon_\alpha} dv_g} \\ t_2 &= \frac{\int_M (\eta_\alpha u_\alpha)^2 dv_\xi - \int_M (\eta_\alpha u_\alpha)^2 dv_g}{\int_M (\eta_\alpha u_\alpha)^2 dv_g} \\ t_\nabla &= \frac{\int_M | \nabla \eta_\alpha u_\alpha |_\xi^2 dv_\xi - \int_M | \nabla \eta_\alpha u_\alpha |_g^2 dv_g}{\int_M | \nabla \eta_\alpha u_\alpha |_g^2 dv_g} \end{aligned}$$

By the Cartan Hadamard expansion of g, we have :

$$dv_\xi = \left(1 + \frac{1}{6} R_{i,j}(x_\alpha)x^i x^j + O(r_\alpha^3) \right) dv_g$$

Coming back to the notations of step 5, we then get :

$$\lim_{\alpha \to 0} \frac{t_1}{A_\alpha} = \lim_{\alpha \to 0} \frac{1}{6} \frac{r_1}{A_\alpha} \tag{51}$$

and :

$$\lim_{\alpha\to 0}\frac{t_2}{A_\alpha}=\lim_{\alpha\to 0}\frac{1}{6}\frac{r_2}{A_\alpha} \tag{52}$$

From (46), we also have :

$$\lim_{\alpha\to 0}\frac{t_\nabla}{A_\alpha}=\lim_{\alpha\to 0}\frac{1}{6}\frac{r_\nabla}{A_\alpha} \tag{53}$$

We write :

$$I_{\xi,\alpha}(u_\alpha\eta_\alpha)-I_{g,\alpha}(u_\alpha\eta_\alpha)=I_{g,\alpha}(u_\alpha\eta_\alpha)\frac{(1+t_\nabla)(1+t_1)^{\frac{4}{n(1+\epsilon_\alpha)}}}{(1+t_2)^{1+\frac{2}{n}}}-I_{g,\alpha}(u_\alpha\eta_\alpha)$$

(50) then follows by (29), (51),(52), (53) and the fact that $\lim_{\alpha\to 0} I_{g,\alpha}(u_\alpha\eta_\alpha)=A_0(n)^{-1}$.

d– *Conclusion*
By Hölder's inequality and Carlen and Loss [3], we have :

$$I_{\xi,\alpha}(\eta_\alpha u_\alpha)\geq\frac{\int_M\mid\nabla\eta_\alpha u_\alpha\mid_\xi^2 dv_\xi\left(\int_M\eta_\alpha u_\alpha dv_\xi\right)^{\frac{4}{n}}}{\int_M(\eta_\alpha u_\alpha)^2 dv_\xi}\geq A_0(n)^{-1}$$

We have then :

$$I_{\xi,\alpha}(\eta_\alpha u_\alpha)-I_{g,\alpha}(\eta_\alpha u_\alpha)\geq A_0(n)^{-1}-I_{g,\alpha}g(\eta_\alpha u_\alpha)$$

Dividing this inequality by A_α and recalling that $B_0=\alpha_0 A_0(n)$, we get from (42) and (50) that :

$$B_0\leq\frac{\mid\mathcal{B}\mid^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2}+\frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}}S_g(x_0)$$

and then :

$$B_0\leq\frac{\mid\mathcal{B}\mid^{-\frac{2}{n}}}{6n}\left(\frac{2}{n+2}+\frac{n-2}{\lambda_1}\right)\left(\frac{n+2}{2}\right)^{\frac{2}{n}}\max_{x\in M}S_g(x)$$

This ends the proof of the theorem.

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